Exercise Sheet #5

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P1. Let $(X, \mathcal{B}, \mu)$ σ-finite measure space. Show that

- **(a)** μ is s-finite.
- **(b)** μ is semi-finite.

P2. Let $(X, \mathcal{B}, \mu)$ be a measure space. Define $\tilde{\mathcal{B}} = \{E \subseteq X \mid \forall F \in \mathcal{B}, \mu(F) < \infty \Rightarrow E \cap F \in \mathcal{B}\}$.

- **(a)** Prove that $\tilde{\mathcal{B}}$ is a sigma algebra.
- **(b)** Define $\tilde{\mu}$ on M by $\tilde{\mu}(E) = \mu(E)$ for $E \in \mathcal{B}$ and $\tilde{\mu}(E) = \infty$ otherwise. Prove that $\tilde{\mu}$ is a saturated measure on $\tilde{\mathcal{B}}$.

P3. Let $\mathcal{P}$ be a π-system that contains X and $\mathcal{F}$ a family of functions from X to $\mathbb{R}$ such that

- **(a)** $A \in \mathcal{P} \Longrightarrow \mathbb{1}_A \in \mathcal{F}$,
- **(b)** $\mathcal{F}$ is a real vector space: $f, g \in \mathcal{F}$ and $c \in \mathbb{R} \Longrightarrow cf + g \in \mathcal{F}$,
- **(c)** if $(f_n)_{n\in\mathbb{N}}$ is a non-decreasing sequence of positive functions in $\mathcal{F}$ and $f = \text{lím}_{n\to\infty} f_n$ bounded, then $f \in \mathcal{F}$.

Show that $\mathcal{F}$ contains the set $\{f : X \to \mathbb{R} \mid f \text{ is a bounded } \sigma(\mathcal{P})\text{-measurable function}\}$.

P4. We will show that if $(X, \mathcal{B}, \mu)$ is a non-atomic probability space, then for all $t \in [0, 1]$, there is $E \in \mathcal{B}$ such that $\mu(E) = t$. For this:

- **(a)** Show that for every $s \in (0, 1)$, there is $E \in \mathcal{B}$ such that $\mu(E) \in (0, s)$.
- **(b)** Fix $t \in (0, 1)$. Construct a family of disjoint sets $(E_n)_{n\in\mathbb{N}} \subseteq \mathcal{B}$ such that:
  - **i)** For each $n \in \mathbb{N}$, $\mu(\bigcup_{i=1}^n E_i) < t$.
  - **ii)** If it is possible, for each $n \in \mathbb{N}$, E_n is chosen such that $\mu(E_n) \geq \frac{1}{n}$.

  Show that $\mu(\bigcup_{n\in\mathbb{N}} E_n) = t$.
  Hint: If the latter is not true, then find $F \in \mathcal{B}$ such that $0 < \mu(F) < t - \mu(\bigcup_{n\in\mathbb{N}} E_n)$. What does this imply for condition **ii)** of the definition of the $(E_n)_{n\in\mathbb{N}}$?

P5. Verify if the following are examples of π-system and/or λ-systems:

- **(a)** The collection $\mathcal{P} = \{(a, b] : a, b \in \mathbb{R}\}$ of half-open intervals in $\mathbb{R}$
- **(b)** Given two measurable spaces $(X, \mathcal{B})$ and $(Y, \mathcal{C})$, the family $\mathcal{P} = \{B \times C : B \in \mathcal{B}, C \in \mathcal{C}\}$ of "rectangles" in $X \times Y$.
- **(c)** For two probability measures μ, ν on a measurable space $(X, \mathcal{B})$, the family $\mathcal{L} = \{E \in \mathcal{B} : \mu(E) = \nu(E)\}$.